

TRANSVERSALITY OF ISOTROPIC PROJECTIONS, UNRECTIFIABILITY AND HEISENBERG GROUPS

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ABSTRACT. We show that the family of m -dimensional isotropic projections in \mathbb{R}^{2n} is transversal. As an application we show that the Besicovitch-Federer projection theorem holds for isotropic projections. We also use transversality to obtain almost sure estimates on the Hausdorff dimension of isotropic projections of subsets $E \subset \mathbb{R}^{2n}$. These results may also be applied to gain information on the horizontal projections of the Heisenberg group \mathbb{H}^n .

1. INTRODUCTION

Let \mathbb{R}^{2n} be equipped with the standard symplectic form $\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}, \omega(x, y) = \sum_{i=1}^n x_{i+n}y_i - x_iy_{i+n}$. In this note we examine properties of m -dimensional isotropic projections in \mathbb{R}^{2n} , that is, orthogonal projections onto m -dimensional isotropic subspaces of \mathbb{R}^{2n} . A linear subspace $V \subset \mathbb{R}^{2n}$ is isotropic if $\omega(v, w) = 0$ for all $v, w \in V$. Isotropic subspaces are closely related to horizontal subgroups of the Heisenberg group \mathbb{H}^n . Indeed, $\mathbb{V} \subset \mathbb{H}^n$ is a horizontal subgroup, if and only if $\mathbb{V} = V \times \{0\}$ for some isotropic subspace $V \subset \mathbb{R}^{2n}$. Thus the study of isotropic projections also yields results on the horizontal projections of \mathbb{H}^n .

Our main result is the following theorem.

Theorem 1.1. *Let n, m be integers such that $0 < m \leq n$, let $G_h(n, m)$ be the submanifold of the Grassmannian $G(2n, m)$ consisting of all isotropic subspaces of \mathbb{R}^{2n} and denote by $P_V : \mathbb{R}^{2n} \rightarrow V$ the orthogonal projection onto the m -plane $V \in G_h(n, m)$. Then the projection family $\{P_V : \mathbb{R}^{2n} \rightarrow V\}_{V \in G_h(n, m)}$ is transversal.*

This paper is organized as follows: In Section 2 we introduce the basic definitions and notation. In Section 3 we will prove Theorem 1.1. As an application of the main result we give necessary and sufficient conditions under which a subset $E \subset \mathbb{R}^{2n}$ projects onto

2000 *Mathematics Subject Classification.* 28A80, 28A78, 53D05.

Key words and phrases. Projection, Symplectic geometry, Heisenberg group, Hausdorff dimension, unrectifiability.

The research was supported by the Finnish Centre of Excellence in Analysis and Dynamics Research and by the Jenny and Antti Wihuri Foundation.

a set of measure zero under almost all m -dimensional isotropic projections (Theorem 4.1). We also give almost sure dimension estimates on the Hausdorff dimension of isotropic projections of subsets of \mathbb{R}^{2n} (Theorem 4.3). These estimates were already proven in [BFMT, Theorem 1.2] using different methods. We improve the result by providing estimates on the dimension of exceptional parameters. The results mentioned above also yield corollaries concerning the dimension of horizontal projections of subsets of the Heisenberg group. The applications will be discussed in Section 4.

2. PRELIMINARIES

2.1. Symplectic geometry. Let M be a manifold of dimension $2n$. A *symplectic form* on M is a closed non-degenerate 2-form on M . The *standard form* ω on \mathbb{R}^{2n} is defined by

$$\omega(x, y) = \sum_{i=1}^n x_{i+n}y_i - x_iy_{i+n} = (Jx | y),$$

where $(\cdot | \cdot)$ is the Euclidean inner product on \mathbb{R}^{2n} and J is the $2n \times 2n$ -matrix

$$J = \begin{pmatrix} 0 & I_{n \times n} \\ -I_{n \times n} & 0 \end{pmatrix}.$$

By a well-known theorem of Darboux, every symplectic form on M is locally diffeomorphic to the standard form ω on \mathbb{R}^{2n} . Furthermore, every symplectic vector space is isomorphic to $(\mathbb{R}^{2n}, \omega)$. Below we work only on \mathbb{R}^{2n} equipped with the standard form ω . For more information on symplectic geometry, see [ET].

For a linear subspace $V \subset \mathbb{R}^{2n}$, we define its *symplectic orthogonal* V^ω by

$$V^\omega = \{w : \omega(w, v) = 0 \text{ for all } v \in V\}.$$

A linear subspace V is said to be *isotropic*, if $V \subset V^\omega$, and *Lagrangian*, if $V = V^\omega$. A subspace V can be Lagrangian only when $\dim V = n$. For integers $0 < m \leq n$, we denote by $G(2n, m)$ the space of all m -dimensional linear subspaces of \mathbb{R}^{2n} . It is a compact manifold of dimension $m(2n - m)$. Furthermore, we define the *isotropic Grassmannian* $G_h(n, m)$ by

$$G_h(n, m) = \{V \in G(2n, m) : V \text{ is an isotropic subspace of } \mathbb{R}^{2n}\}.$$

In the case $m = n$, $G_h(n, n)$ is called the *Lagrangian Grassmannian*. $G_h(n, m)$ is a smooth manifold of dimension $2nm - \frac{m(3m-1)}{2}$, so for $m > 1$ the isotropic Grassmannian $G_h(n, m)$ is a submanifold of $G(2n, m)$ with positive codimension. For $m = 1$, the manifolds are the same, $G_h(n, 1) = G(2n, 1)$. The isotropic Grassmannian can be endowed with a natural measure $\mu_{n,m}$ in a similar way as the usual Grassmannian is endowed with the measure $\gamma_{n,m}$, using unitary instead of orthogonal matrices. See [BFMT, Section 2] for more details.

Next we define local coordinates on the isotropic Grassmannian $G_h(n, m)$. We begin by recalling the definition of local coordinates on the Grassmannian manifold $G(2n, m)$. Fix an m -plane $V \in G(2n, m)$ and choose an orthonormal basis $\{e_1, \dots, e_m\}$ of \mathbb{R}^{2n} such that $V = \langle e_1, \dots, e_m \rangle$. Consider all linear maps $L(V, V^\perp) = \{L : V \rightarrow V^\perp : L \text{ linear}\}$. The graph $\mathcal{G}(L) = \{(x, Lx) : x \in V\}$ of any such map is an m -plane whose intersection with the $(2n - m)$ -plane V^\perp is the zero subspace. Conversely, any m -plane with this property is the graph of a unique linear map $L : V \rightarrow V^\perp$. Using the basis $\{e_1, \dots, e_m\}$ of V and the basis $\{e_{m+1}, \dots, e_{2n}\}$ for V^\perp , $L(V, V^\perp)$ can be identified with $\mathcal{M}(2n - m, m)$, the space of all $(2n - m) \times m$ matrices. The m -plane associated to a matrix $A = (a_{ij}) \in \mathcal{M}(2n - m, m)$ is spanned by the vectors

$$(2.1) \quad e_i^A = e_i + \sum_{k=1}^{2n-m} a_{ki} e_{k+m}, \quad i = 1, \dots, m.$$

Define a subset $\mathcal{M}_h(n, m)$ of all $(2n - m) \times m$ matrices by setting

$$(2.2) \quad \begin{aligned} \mathcal{M}_h(n, m) = \{ (a_{ij}) \in \mathcal{M}(2n - m, m) : a_{(n-m+i)j} = a_{(n-m+j)i} \\ + \sum_{k=1}^{n-m} (a_{kj} a_{(n+k)i} - a_{(n+k)j} a_{ki}) \text{ for } j < i \leq m, 1 \leq j \leq m \}. \end{aligned}$$

The independent coordinates a_{ij} in a matrix $(a_{ij}) \in \mathcal{M}_h(n, m)$ are the ones with $j \in \{1, \dots, m\}$, $i \in \{1, \dots, n - m + j, n + 1, \dots, 2n - m\}$.

$\mathcal{M}_h(n, m)$ is an embedded submanifold of $\mathcal{M}(2n - m, m)$ having dimension $2nm - \frac{m(3m-1)}{2}$. Let $A \in \mathcal{M}(2n - m, m)$ and denote the m -plane associated to A by V_A . Then $V_A \in G_h(n, m)$ if and only if

$$\omega(e_i + \sum_{k=1}^{2n-m} a_{ki} e_{k+m}, e_j + \sum_{k=1}^{2n-m} a_{kj} e_{k+m}) = 0 \text{ for all } i, j = 1, \dots, m$$

and this is the case precisely when $A \in \mathcal{M}_h(n, m)$. Thus we can define coordinates on the isotropic Grassmannian using the matrices $A \in \mathcal{M}_h(n, m)$. It is enough to define the local coordinates only around the m -plane $V = \langle e_1, \dots, e_m \rangle$, since the group $U(n)$ of unitary $2n \times 2n$ -matrices acts transitively on $G_h(n, m)$ by [BFMT, Lemma 2.2] and $\omega(gu, gv) = \omega(u, v)$ for all $u, v \in \mathbb{R}^{2n}$, $g \in U(n)$.

2.2. Heisenberg groups. For an introduction to Heisenberg groups, see [CDPT]. Below we state the basic facts needed in this paper. The Heisenberg group \mathbb{H}^n is the unique simply connected, connected nilpotent Lie group of step two and dimension $2n + 1$ with one dimensional centre. As a manifold \mathbb{H}^n may be identified with \mathbb{R}^{2n+1} . We denote points $p \in \mathbb{H}^n$ in coordinates as

$$p = (z, t) = (z_1, \dots, z_{2n}, t) \in \mathbb{R}^{2n} \times \mathbb{R}.$$

The group operation is given by

$$p * p' = (z, t) * (z', t') = (z + z', t + t' + 2\omega(z, z')),$$

where ω is the standard symplectic form on \mathbb{R}^{2n} .

The Heisenberg metric d_H of \mathbb{H}^n can be defined by

$$d_H(p, p') := \|p^{-1} * p'\|_H, \quad \text{where } \|p\|_H := (\|z\|^4 + t^2)^{1/4}.$$

Here $\|\cdot\|$ denotes the Euclidean norm on \mathbb{R}^{2n} . This metric is bi-Lipschitz equivalent to the usual Carnot-Caratheodory metric on \mathbb{H}^n . The metric d_H induces the Euclidean topology, but the properties of the metric space (\mathbb{H}^n, d_H) differ significantly from those of the underlying Euclidean space. For example, the Hausdorff dimension of (\mathbb{H}^n, d_H) is $2n+2$. Thus, when speaking of the metric properties of \mathbb{H}^n , we need to specify which metric we are using. We will denote the Hausdorff measure and Hausdorff dimension with respect to the Heisenberg metric by \mathcal{H}_H^s and \dim_H . The Hausdorff measure and dimension with respect to the Euclidean metric are denoted by \mathcal{H}_E^s and \dim_E .

In this paper we consider projections onto homogeneous subgroups of \mathbb{H}^n . A *homogeneous subgroup* \mathbb{G} of \mathbb{H}^n is a subgroup which is closed under the intrinsic dilatations $\delta_s(z, t) = (sz, s^2t)$, $s > 0$. There are two kinds of homogeneous subgroups of \mathbb{H}^n . The *horizontal subgroups* are the ones which are contained in $\mathbb{R}^{2n} \times \{0\}$ and the *vertical subgroups* are the ones which contain the t -axis $\{0\} \times \mathbb{R}$. Horizontal subgroups can be identified with linear subspaces of \mathbb{R}^{2n+1} which are contained in $\mathbb{R}^{2n} \times \{0\}$. However not every linear subspace of this form is a horizontal subgroup, only those corresponding to isotropic subspaces V of \mathbb{R}^{2n} . The restriction of the Heisenberg metric to a horizontal subgroup coincides with the Euclidean metric and therefore it is not necessary to specify the metric used in computing the Hausdorff measure or Hausdorff dimension of a subset of a horizontal subgroup. In this case we denote the Hausdorff measure and Hausdorff dimension simply by \mathcal{H}^s and \dim .

Let $\mathbb{V} = V \times \{0\}$ be a horizontal subgroup. Consider $\mathbb{V}^\perp = V^\perp \times \mathbb{R}$, where V^\perp is the orthogonal complement of V in \mathbb{R}^{2n} . Then \mathbb{V}^\perp is a vertical subgroup of \mathbb{H}^n and it will be called the vertical subgroup associated to V . Each point $p \in \mathbb{H}^n$ can be written uniquely as

$$p = P_{\mathbb{V}^\perp}(p) * P_{\mathbb{V}}(p),$$

with $P_{\mathbb{V}^\perp}(p) \in \mathbb{V}^\perp$ and $P_{\mathbb{V}}(p) \in \mathbb{V}$. This gives rise to a well-defined *horizontal projection*

$$P_{\mathbb{V}} : \mathbb{H}^n \rightarrow \mathbb{V}, \quad (z, t) \mapsto P_{\mathbb{V}}(z, t) = (P_V(z), 0),$$

and a *vertical projection*

$$P_{\mathbb{V}^\perp} : \mathbb{H}^n \rightarrow \mathbb{V}^\perp, \quad (z, t) \mapsto P_{\mathbb{V}^\perp}(z, t) = (P_{V^\perp}(z), t - 2\omega(P_{V^\perp}(z), P_V(z))).$$

Since there is a one-to-one correspondence between isotropic subspaces of \mathbb{R}^{2n} and horizontal subgroups of \mathbb{H}^n , these projections may be parametrized by the isotropic

Grassmannian $G_h(n, m)$. For more information on the projections of the Heisenberg group, see [BCFMT] and [BFMT].

We denote the family of all horizontal projections onto m -dimensional subgroups by $\mathcal{F}_h(n, m)$ and the corresponding projections in the Euclidean space \mathbb{R}^{2n} by $\mathcal{F}_h^{2n}(n, m)$, that is,

$$\mathcal{F}_h(n, m) = \{P_V : \mathbb{H}^n \rightarrow \mathbb{V} : V \in G_h(n, m)\}$$

and

$$\mathcal{F}_h^{2n}(n, m) = \{P_V : \mathbb{R}^{2n} \rightarrow V : V \in G_h(n, m)\}.$$

Note that when $m > 1$, the family $\mathcal{F}_h^{2n}(n, m)$ has dimension $2nm - \frac{m(3m-1)}{2} < m(2n - m)$, and therefore one cannot apply standard projection theorems (e.g. Marstrand's projection theorem or Besicovitch-Federer projection theorem) to obtain dimension results for these projections.

3. TRANSVERSALITY

In this section we show that the family $\mathcal{F}_h^{2n}(n, m)$ of projections is transversal for every $0 < m \leq n$. We begin with the definition of transversality.

Definition 3.1. Let $\Lambda \subset \mathbb{R}^l$ be open. A family of maps $\{\pi_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m\}_{\lambda \in \Lambda}$ is *transversal* if it satisfies the following conditions for each compact set $K \subset \mathbb{R}^n$:

- (1) The mapping $\pi : \Lambda \times K \rightarrow \mathbb{R}^m$, $(\lambda, x) \mapsto \pi_\lambda(x)$, is continuously differentiable and twice differentiable with respect to λ .
- (2) For $j = 1, 2$ there exist constants C_j such that the derivatives with respect to λ satisfy

$$\|D_\lambda^j \pi(\lambda, x)\| \leq C_j \text{ for all } (\lambda, x) \in \Lambda \times K.$$

- (3) For all $\lambda \in \Lambda$ and $x, y \in K$ with $x \neq y$, define

$$\Phi_{x,y}(\lambda) = \frac{\pi_\lambda(x) - \pi_\lambda(y)}{\|x - y\|}.$$

Then there exists a constant $C_T > 0$ such that the property

$$\|\Phi_{x,y}(\lambda)\| \leq C_T$$

implies that

$$\det \left(D_\lambda \Phi_{x,y}(\lambda) (D_\lambda \Phi_{x,y}(\lambda))^T \right) \geq C_T^2.$$

- (4) There exists a constant C_L such that

$$\|D_\lambda^2 \Phi_{x,y}(\lambda)\| \leq C_L$$

for all $\lambda \in \Lambda$ and $x, y \in K$, $x \neq y$.

Since transversality is a local property and the isotropic Grassmannian $G_h(n, m)$ can be covered by finitely many coordinate neighbourhoods, we need to show that the projection family $\mathcal{F}_h^{2n}(n, m)$ satisfies the above conditions locally, that is, each $V \in G_h(n, m)$ has a coordinate neighbourhood $U \subset G_h(n, m)$ on which $\{P_W : \mathbb{R}^{2n} \rightarrow W\}_{W \in U}$ is transversal.

Transversal projection families have many useful properties. For instance, the Hausdorff dimensions of sets and measures are preserved under almost all projections. The theory of transversal mappings was extensively studied by Peres and Schlag in [PS]. A recent result for transversal projection families that we will use in this paper is the Besicovitch-Federer projection theorem. See [HJJL] for the proof.

Theorem 3.2. *Let $E \subset \mathbb{R}^n$ be \mathcal{H}^m -measurable with $\mathcal{H}^m(E) < \infty$. Assume that $\Lambda \subset \mathbb{R}^l$ is open and $\{\pi_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m\}_{\lambda \in \Lambda}$ is a transversal family of maps. Then E is purely m -unrectifiable, if and only if $\mathcal{H}^m(\pi_\lambda(E)) = 0$ for \mathcal{L}^l -almost all $\lambda \in \Lambda$.*

Fix $0 < m \leq n$. To show that the transversality condition holds locally, we fix an m -plane $V \in G_h(n, m)$, take a coordinate system around V and show that the transversality conditions hold in this coordinate neighbourhood.

Define a family of projections $\pi : \mathcal{M}_h(n, m) \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^m$ by setting

$$\begin{aligned} \pi(A, x) &= \pi_A(x) = \left(\left(e_1^A \middle| x \right), \dots, \left(e_m^A \middle| x \right) \right) \\ &= \left(x_1 + \sum_{k=1}^{2n-m} a_{k1} x_{k+m}, \dots, x_m + \sum_{k=1}^{2n-m} a_{km} x_{k+m} \right). \end{aligned}$$

The projection π_A is not quite the same as the orthogonal projection P_{V_A} onto the m -plane V_A corresponding to the matrix A , since the base $\{e_1^A, \dots, e_m^A\}$ of V_A is not orthonormal, but we will see that for A close to 0 the projections are sufficiently close.

Define a mapping $\Phi : \{(A, x, y) \in \mathcal{M}_h(n, m) \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} : x \neq y\} \rightarrow \mathbb{R}^m$ by

$$\Phi(A, x, y) = \Phi_{x,y}(A) = \frac{\pi_A(x) - \pi_A(y)}{\|x - y\|}.$$

Denoting $b = \|x - y\|^{-1}(x - y) \in S^{2n-1}$ and using the linearity of the projection π_A we see that

$$\Phi_{x,y}(A) = \frac{\pi_A(x) - \pi_A(y)}{\|x - y\|} = \frac{\pi_A(x - y)}{\|x - y\|} = \pi_A \left(\frac{x - y}{\|x - y\|} \right) = \pi_A(b) =: \Phi_b(A).$$

We will show that for the family defined above it holds that for every $x, y \in \mathbb{R}^{2n}$ with $x \neq y$,

$$(3.1) \quad \det \left(D_A \Phi_{x,y}(0) (D_A \Phi_{x,y}(0))^T \right) \geq \frac{1}{2} (1 - \|\Phi_{x,y}(0)\|^2)^m.$$

Before we prove this inequality, we show that it implies that the family $\mathcal{F}_h^{2n}(n, m)$ is locally transversal. The family clearly satisfies the conditions (1), (2) and (4) in Definition 3.1, so we need to show that the condition (3) is also valid. We examine the problem in local coordinates (U, φ) defined above. Let $\{v_1^A, \dots, v_m^A\}$ be the orthonormal basis obtained by applying the Gram-Schmidt algorithm to the basis $\{e_1^A, \dots, e_m^A\}$. Then at $A = 0$, we have for every $b \in S^{2n-1}$

$$(3.2) \quad (v_i^0|b) = (e_i^0|b)$$

and

$$(3.3) \quad \partial_{\alpha\beta}|_{A=0} (v_i^A|b) = \partial_{\alpha\beta}|_{A=0} (e_i^A|b)$$

for all $i, \beta = 1, \dots, m$, and $\alpha \in \{1, \dots, n-m+\beta, n+1, \dots, 2n-m\}$, where $\partial_{\alpha\beta}$ denotes the partial derivative with respect to the entry $a_{\alpha\beta}$ in the matrix $A = (a_{\alpha\beta})$.

Defining $\Phi^{\mathcal{F}} : \{(A, x, y) \in \mathcal{M}_h(n, m) \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} : x \neq y\} \rightarrow \mathbb{R}^m$ by

$$\Phi^{\mathcal{F}}(A, x, y) = \Phi_{x,y}^{\mathcal{F}}(V_A) = \frac{P_{V_A}(x) - P_{V_A}(y)}{\|x - y\|},$$

equations (3.2) and (3.3) imply that

$$\Phi_{x,y}^{\mathcal{F}}(V_0) = \Phi_{x,y}(0)$$

and

$$\det \left(D_A \Phi_{x,y}(0) (D_A \Phi_{x,y}(0))^T \right) = \det \left(D_A \Phi_{x,y}^{\mathcal{F}}(V_0) (D_A \Phi_{x,y}^{\mathcal{F}}(V_0))^T \right)$$

for every $x, y \in \mathbb{R}^{2n}$ such that $x \neq y$. We also use the notation $\tilde{\Phi}^{\mathcal{F}}(A, b) = \tilde{\Phi}_b^{\mathcal{F}}(V_A) = P_{V_A}(b)$ for $b \in S^{2n-1}$. Note that $\tilde{\Phi}^{\mathcal{F}}(A, (x - y)/\|x - y\|) = \Phi^{\mathcal{F}}(A, x, y)$. The functions $\Phi^{\mathcal{F}}$ and $\tilde{\Phi}^{\mathcal{F}}$ are smooth, so defining $\mathcal{M}_h^1(n, m) = \mathcal{M}_h(n, m) \cap B(0, 1)$, we may choose a Lipschitz constant $L_1 \geq 1$ for $\tilde{\Phi}^{\mathcal{F}}$ and a Lipschitz constant $L_2 \geq 1$ for

$$(A, b) \mapsto \det \left(D_A \tilde{\Phi}_b^{\mathcal{F}}(V_A) \left(D_A \tilde{\Phi}_b^{\mathcal{F}}(V_A) \right)^T \right) \text{ on } \mathcal{M}_h^1(n, m) \times S^{2n-1}.$$

Let

$$0 < C_T \leq 2^{-\frac{m+2}{2}} \quad \text{and} \quad \epsilon = \min \left\{ \frac{C_T}{L_1}, \frac{(1 - 4C_T^2)^m}{4L_2} \right\}.$$

If $(A, b) \in (\mathcal{M}_h(n, m) \cap B(0, \epsilon)) \times S^{2n-1}$ is such that

$$\|\tilde{\Phi}_b^{\mathcal{F}}(V_A)\| \leq C_T,$$

we have by (3.2)

$$\|\Phi_b(0)\| = \|\tilde{\Phi}_b^{\mathcal{F}}(V_0)\| \leq L_1 \epsilon + C_T \leq 2C_T$$

and by (3.1) and (3.3)

$$\begin{aligned} \det \left(D_A \tilde{\Phi}_b^{\mathcal{F}}(V_0) \left(D_A \tilde{\Phi}_b^{\mathcal{F}}(V_0) \right)^T \right) &= \det \left(D_A \Phi_b(0) (D_A \Phi_b(0))^T \right) \\ &\geq \frac{1}{2} (1 - \|\Phi_b(0)\|^2)^m \\ &\geq \frac{1}{2} (1 - 4C_T^2)^m. \end{aligned}$$

This implies that

$$\det \left(D_A \tilde{\Phi}_b^{\mathcal{F}}(V_A) \left(D_A \tilde{\Phi}_b^{\mathcal{F}}(V_A) \right)^T \right) \geq \frac{1}{2} (1 - 4C_T^2)^m - L_2 \epsilon \geq \frac{1}{4} (1 - 4C_T^2)^m \geq C_T^2$$

by the choice of C_T and ϵ . We have shown that assuming inequality (3.1), every plane $V \in G_h(n, m)$ has a coordinate neighbourhood on which the family $\mathcal{F}_h^{2n}(n, m)$ satisfies the transversality condition. Next we prove the inequality (3.1).

Recalling (2.2), we see that for $A \in \mathcal{M}_h(n, m)$ the j th component function of the projection π_A has the form

$$\begin{aligned} \pi_A^j(x) &= x_j + \sum_{k=1}^{2n-m} a_{kj} x_{m+k} \\ &= x_j + \sum_{k=1}^{n-m+j} a_{kj} x_{m+k} + \sum_{k=n+1}^{2n-m} a_{kj} x_{m+k} \\ &\quad + \sum_{k=j+1}^m \left(a_{(n-m+j)k} + \sum_{l=1}^{n-m} (a_{lj} a_{(l+n)k} - a_{lk} a_{(l+n)j}) \right) x_{n+k}. \end{aligned}$$

The partial derivatives with respect to the entries in the matrix A are

$$\partial_{\alpha\beta} \pi_A^j(x) = \begin{cases} x_{m+\alpha} + \sum_{k=j+1}^m a_{(\alpha+n)k} x_{n+k}, & \text{for } \beta = j, 1 \leq \alpha \leq n-m \\ x_{m+\alpha}, & \text{for } \beta = j, n-m+1 \leq \alpha \leq n-m+j \\ x_{m+\alpha} - \sum_{k=j+1}^m a_{(\alpha-n)k} x_{n+k}, & \text{for } \beta = j, n+1 \leq \alpha \leq 2n-m \\ -a_{(\alpha+n)j} x_{n+\beta}, & \text{for } \beta > j, 1 \leq \alpha \leq n-m \\ x_{n+\beta}, & \text{for } \beta > j, \alpha = n-m+j \\ a_{(\alpha-n)j} x_{n+\beta}, & \text{for } \beta > j, n+1 \leq \alpha \leq 2n-m \\ 0, & \text{elsewhere.} \end{cases}$$

From this we may compute the matrix $B_{A,x} = D_A \pi_A(x) (D_A \pi_A(x))^T$ at $A = 0$: The matrix $D_A \pi_A(x)$ is an $m \times (2nm - \frac{m(3m-1)}{2})$ -matrix. The rows correspond to the component functions of the mapping π_A and the columns correspond to all possible pairs (α, β) . When $A = 0$, the entries on the j th row of the matrix $D_A \pi_A(x)$ are

$$\begin{aligned}
& x_{m+\alpha}, & \text{for } \beta = j, \alpha \in \{1, \dots, n-m+j, n+1, \dots, 2n-m\} \\
& x_{n+\beta}, & \text{for } \alpha = n-m+j, \beta \in \{j+1, \dots, m\} \\
& 0, & \text{elsewhere.}
\end{aligned}$$

From this we see that the coordinates x_{m+1}, \dots, x_{2n} appear on each row exactly once and the other entries are zero. Thus all the diagonal entries of the matrix $B_{0,x}$ are $[B_{0,x}]_{ii} = \sum_{k=1}^{2n-m} x_{m+k}^2$ for every $i = 1, \dots, m$.

The entries x_{m+1}, \dots, x_{2n} appear on different positions on different rows. If $i < j$, there exists a non-zero entry at the same column on both rows i and j if and only if $\alpha = n-m+i$ and $\beta = j$. The entry on i th row is $x_{n+\beta} = x_{n+j}$ and the entry on j th row is $x_{m+\alpha} = x_{n+i}$. This implies that the non-diagonal entries of the matrix $B_{0,x}$ are $[B_{0,x}]_{ij} = x_{n+i}x_{n+j}$ for every $i, j = 1, \dots, m, i \neq j$. Thus

$$[B_{0,x}]_{ij} = \begin{cases} \sum_{k=1}^{2n-m} x_{m+k}^2 =: \Delta_x, & \text{for } i = j \\ x_{n+i}x_{n+j} & \text{for } i \neq j. \end{cases}$$

Using an inductive argument one can see that the determinant of such matrix is

$$\det B_{0,x} = \Delta_x^m + \sum_{i=2}^m (-1)^{i-1} (i-1) \Delta_x^{m-i} \sum_{\alpha \in \Lambda(m,i)} x_{\alpha(1)}^2 \cdots x_{\alpha(i)}^2,$$

where

$$\Lambda(m, i) = \{\alpha = (\alpha(1), \dots, \alpha(i)) : \alpha(k) \in \{n+1, \dots, n+m\} \forall k, \alpha(1) < \dots < \alpha(i)\}$$

is the set of all strictly increasing sequences of length i consisting of integers from the interval $[n+1, n+m]$.

Let $b \in S^{2n-1}$. Note that the entry Δ_b in the diagonal of the matrix $B_{0,b}$ is precisely $\|P_{V_0^\perp}(b)\|^2 = 1 - \|\Phi_b(0)\|^2$. We will show that

$$(3.4) \quad \det B_{0,b} \geq \Delta_b^m - \Delta_b^{m-2} \sum_{\alpha \in \Lambda(m,2)} b_{\alpha(1)}^2 b_{\alpha(2)}^2 \geq \frac{1}{2} \Delta_b^m.$$

The second inequality is easy:

$$\begin{aligned}
\frac{1}{2} \Delta_b^m - \Delta_b^{m-2} \sum_{\alpha \in \Lambda(m,2)} b_{\alpha(1)}^2 b_{\alpha(2)}^2 &\geq \frac{1}{2} \Delta_b^{m-2} \left(\left(\sum_{i=1}^m b_{n+i}^2 \right)^2 - \sum_{\alpha \in \Lambda(m,2)} 2b_{\alpha(1)}^2 b_{\alpha(2)}^2 \right) \\
&= \frac{1}{2} \Delta_b^{m-2} \sum_{i=1}^m b_{n+i}^4 \\
&\geq 0.
\end{aligned}$$

For the first inequality it is enough to show that for any $i \in \{3, \dots, m-1\}$ it holds that

$$(i-1)\Delta_b^{m-i} \sum_{\alpha \in \Lambda(m,i)} b_{\alpha(1)}^2 \cdots b_{\alpha(i)}^2 \geq i\Delta_b^{m-i-1} \sum_{\alpha \in \Lambda(m,i+1)} b_{\alpha(1)}^2 \cdots b_{\alpha(i+1)}^2.$$

Using the fact that $\Delta_b \geq \sum_{j=n+1}^{n+m} b_j^2$, we see that the above inequality holds if

$$(i-1) \sum_{\alpha \in \Lambda(m,i)} b_{\alpha(1)}^2 \cdots b_{\alpha(i)}^2 \left(\sum_{j=n+1}^{n+m} b_j^2 \right) \geq i \sum_{\alpha \in \Lambda(m,i+1)} b_{\alpha(1)}^2 \cdots b_{\alpha(i+1)}^2.$$

Now

$$\begin{aligned} & (i-1) \sum_{\alpha \in \Lambda(m,i)} b_{\alpha(1)}^2 \cdots b_{\alpha(i)}^2 \left(\sum_{j=n+1}^{n+m} b_j^2 \right) - i \sum_{\alpha \in \Lambda(m,i+1)} b_{\alpha(1)}^2 \cdots b_{\alpha(i+1)}^2 \\ &= (i-1) \sum_{\alpha \in \Lambda(m,i)} b_{\alpha(1)}^2 \cdots b_{\alpha(i)}^2 \left(\sum_{j=n+1}^{\alpha(i)} b_j^2 \right) - \sum_{\alpha \in \Lambda(m-1,i)} b_{\alpha(1)}^2 \cdots b_{\alpha(i)}^2 \left(\sum_{k=\alpha(i)+1}^{n+m} b_k^2 \right) \\ &= (i-1) \sum_{k=n+i}^{n+m} \sum_{j=n+1}^k \sum_{\alpha \in \Lambda(k-1,i-1)} b_{\alpha(1)}^2 \cdots b_{\alpha(i-1)}^2 b_j^2 b_k^2 \\ &\quad - \sum_{j=n+i}^{n+m-1} \sum_{k=j+1}^{n+m} \sum_{\alpha \in \Lambda(j-1,i-1)} b_{\alpha(1)}^2 \cdots b_{\alpha(i-1)}^2 b_k^2 b_j^2 \\ &\geq \sum_{k=n+i+1}^{n+m} b_k^2 \left(\sum_{j=n+1}^k \sum_{\alpha \in \Lambda(k-1,i-1)} b_{\alpha(1)}^2 \cdots b_{\alpha(i-1)}^2 b_j^2 \right. \\ &\quad \left. - \sum_{j=n+i}^{k-1} \sum_{\alpha \in \Lambda(j-1,i-1)} b_{\alpha(1)}^2 \cdots b_{\alpha(i-1)}^2 b_j^2 \right) \\ &\geq 0, \end{aligned}$$

which proves the first inequality in (3.4). This shows that the inequality (3.1) is valid and finishes the proof that the family $\mathcal{F}_h^{2n}(n, m)$ is transversal.

4. APPLICATIONS

Transversality together with Theorem 3.2 imply that the Besicovitch-Federer projection theorem holds for isotropic projections.

Theorem 4.1. *Let $E \subset \mathbb{R}^{2n}$ be \mathcal{H}^m -measurable with $\mathcal{H}^m(E) < \infty$. Then E is purely m -unrectifiable, if and only if $\mathcal{H}^m(P_V(E)) = 0$ for $\mu_{n,m}$ -almost all $V \in G_h(n, m)$.*

The α -energy $I_\alpha(\mu)$ of a measure μ on \mathbb{R}^n is defined by

$$I_\alpha(\mu) = \int \int |x - y|^{-\alpha} d\mu y d\mu x$$

and the Sobolev dimension of a finite measure μ on \mathbb{R}^n is defined as

$$\dim_S \mu = \sup \left\{ \alpha \in \mathbb{R} : \int (1 + |x|)^{\alpha-n} |\hat{\mu}(x)|^2 dx < \infty \right\},$$

where $\hat{\mu}$ is the Fourier transform of the measure μ .

Transversality immediately yields also the following result concerning the dimension of projected measures. See [PS, Theorem 7.3] for more information.

Theorem 4.2. *Let $0 < m \leq n$ and suppose that μ is a finite positive measure on \mathbb{R}^{2n} with finite α -energy for some $\alpha > 0$. If $\sigma \in (0, \alpha]$, then*

$$\dim\{V \in G_h(n, m) : \dim_S \mu_V < \sigma\} \leq 2nm - \frac{m(3m-1)}{2} + \sigma - \max\{\alpha, m\},$$

where μ_V is the projection of the measure μ onto the m -plane V , that is, $\mu_V(A) = \mu(P_V^{-1}(A))$ for all $A \subset V$.

It follows from the definition of the Sobolev dimension that if $0 < \dim_S \mu \leq n$, then $\dim_S \mu = \sup\{\alpha : I_\alpha(\mu) < \infty\}$. In particular, if a Borel set $E \subset \mathbb{R}^n$ supports a probability measure μ with $\dim_S(\mu) \leq n$, then $\dim E \geq \dim_S \mu$. If $\dim_S \mu > n$, then μ is absolutely continuous. These facts together with Theorem 4.2 imply the following result on the dimension of exceptional sets.

Theorem 4.3. *Let n, m be integers such that $0 < m \leq n$ and let $E \subset \mathbb{R}^{2n}$ be a Borel set with $\dim E = s$.*

- (1) *If $s \leq m$, $\dim\{V \in G_h(n, m) : \dim P_V(E) < s\} \leq 2nm - \frac{m(3m+1)}{2} + s$.*
- (2) *If $s > m$, $\dim\{V \in G_h(n, m) : \mathcal{H}^m(P_V(E)) = 0\} \leq 2nm - \frac{3m(m-1)}{2} - s$.*

Proof. Assume first that $s > m$. Then by Frostman's lemma we can take $\alpha > m$ and a probability measure μ supported on E such that $I_\alpha(\mu) < \infty$. Now Theorem 4.2 implies that

$$\begin{aligned} & \dim\{V \in G_h(n, m) : \mu_V \text{ is not absolutely continuous} \} \\ & \leq \dim\{V \in G_h(n, m) : \dim_S \mu_V \leq m\} \leq 2nm - \frac{3m(m-1)}{2} - \alpha. \end{aligned}$$

It follows that

$$\dim\{V \in G_h(n, m) : \mathcal{H}^m(P_V(E)) = 0\} \leq 2nm - \frac{3m(m-1)}{2} - \alpha.$$

Letting $\alpha \nearrow s$ implies the claim. The first claim is proven similarly. \square

In [BFMT, Theorem 1.2] it is shown that almost all isotropic projections onto m -planes preserve the Hausdorff dimension for sets whose dimension is at most m . For sets with dimension greater than m almost all projections have positive \mathcal{H}^m measure. Their proof uses energy estimates and Frostman's lemma. Theorem 4.3 strengthens this theorem by providing dimension estimates for the sets of exceptional parameters.

The behaviour of dimensions of sets and measures under subfamilies of orthogonal projections has recently been studied by E. Järvenpää, M. Järvenpää and T. Keleti in [JJK] and D. Oberlin in [O]. Their results, however, do not give anything new to our setting due to the fact that the family we are studying is transversal.

Theorems 4.1 and 4.3 yield corresponding results for the horizontal projections in the Heisenberg group. We denote by $\pi : \mathbb{H}^n \rightarrow \mathbb{R}^{2n}$, $\pi(z, t) = z$ the projection onto the first $2n$ coordinates.

Corollary 4.4. *Let $E \subset \mathbb{H}^n$ be a Borel set with $\mathcal{H}_E^m(\pi(E)) < \infty$. Then $\mathcal{H}^m(P_V(E)) = 0$ for $\mu_{n,m}$ -almost all $V \in G_h(n, m)$, if and only if $E \subset A \times \mathbb{R}$, where $A \subset \mathbb{R}^{2n}$ is purely m -unrectifiable in the Euclidean sense.*

Corollary 4.5. *Let n, m be integers such that $0 < m \leq n$ and let $E \subset \mathbb{H}^n$ be a Borel set with $\dim_{\mathbb{H}} E = s$.*

- (1) *If $s \leq m + 2$, $\dim\{V \in G_h(n, m) : \dim P_V(E) < s - 2\} \leq 2nm - \frac{m(3m+1)}{2} + s - 2$.*
- (2) *If $s > m + 2$, $\dim\{V \in G_h(n, m) : \mathcal{H}^m(P_V(E)) = 0\} \leq 2nm - \frac{3m(m-1)}{2} - s + 2$.*

Proof. Since $P_V = P_V \circ \pi$, we have by [BFMT, Proof of Theorem 1.1] that $\dim_E \pi(E) \geq \dim_{\mathbb{H}} E - 2 = s - 2$. The rest of the proof is the same as in Theorem 4.3. \square

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